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NOTE

ON THE SUPPORT SIZE OF NULL DESIGNS OF FINITE RANKED POSETS

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Classically, null designs are defined on Boolean algebras. Null designs can be defined on arbitrary finite ranked posets in an obvious way. A lower bound for the support size of non-zero null designs is given, which proves the conjecture of G. James about the support size of null t-designs of the lattice of subspaces of a vector space over a finite field. The lower bound we find gives the tight bound for many important posets including the Boolean algebra, the lattice of subspaces of a vector space over a finite field, whereas the idea of the proofs of the main theorems makes it possible to prove that the lower bounds in the main theorems are not tight for some posets.

1. Introduction

Let P be a finite ranked partially ordered set. Given two ranks of P, $t \le k$, we can form a 0,1 matrix, called an adjacency matrix, with columns indexed by the elements of rank k and the rows indexed by the elements of rank t. The kernel of this adjacency matrix forms a space of very interesting objects called *null t-designs* [2]. In this paper, we consider the space of null t-designs of finite ranked posets.

We are especially interested in the number of non-zero entries, called the *support size*, of non-zero null t-designs. P. Frankl and J. Pach [1] proved that the minimum support size of non-zero null t-designs is 2^{t+1} for the Boolean algebras. G. James [3, p.114] made a conjecture that $2(1+q)\cdots(1+q^t)$ is the minimum support size of non-zero null t-designs of the lattices of subspaces over a finite field, if $k=t+1$.

By generalizing the idea of P. Frankl and J. Pach, we prove two general theorems (Theorem 3.3 and Theorem 3.4) giving lower bounds for the support size of non-zero null t-designs of finite ranked posets with appropriate properties. The bounds given in the main theorems turn out to be tight for many important posets including the Boolean algebra, the lattice of subspaces of a vector space over

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a finite field and the lattice of Hamming scheme. In particular, Theorem 3.4 proves the conjecture of G. James.

In section 2, we give the definition of *null t-designs* of a finite ranked poset with some special properties. In section 3, we prove the main theorems and their corollaries. In section 4, we give some examples for which the bounds in the main theorems are tight. Finally, in section 5, we make some remarks on related problems.

2. Null designs

Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{R}[X] \equiv \left\{\sum_{x\in X} c_x x : c_x \in \mathbb{R}\right\}$ denote the vector space over $\mathbb{R}$ with a basis X, for a given finite set X. If P is a finite ranked poset, let X_i^P be the set of elements of rank i of P. For $j \leq i$, let

$$d_{i,j}^P : \mathbb{R}[X_i^P] \to \mathbb{R}[X_j^P]$$

be the linear map defined on a basis by

$$\text{for} \quad x \in X_i^P, \quad d_{i,j}^P(x) = \sum_{\substack{y \leq x \\ y \in X_j^P}} y.$$

Definition 2.1. For integers $0 \leq t < k$, a *null* (t, k)*-design* (*null t-design* if there is no confusion) of a finite ranked poset P is an element of the kernel of $d_{k,t}^P$.

Notation 2.2. We will use $N_P(t,k)$ for the vector space of null (t,k)-designs of P.

Definition 2.3. For a finite ranked poset P, we say that

1. P satisfies the *strong downmap condition for the level l* (SD_l) if the condition

$$c_{t',t,k} d_{k,t'}^P = d_{t,t'}^P \circ d_{k,t}^P, \quad c_{t',t,k} \in \mathbb{R}$$

holds for all $t' \leq t < k \leq l$, where l is fixed.

2. P satisfies the *downmap condition for the level l* (D_l), l is fixed, if

$$d_{k,t}^P(x) = 0 \quad \text{implies} \quad d_{k,t'}^P(x) = 0 \quad \text{for } t' \leq t, \quad \text{for all } t < k \leq l.$$

Note that if P satisfies (SD_l), then it automatically satisfies (D_l). The condition (D_l) is equivalent to say that "for $t' \leq t < k \leq l$, null (t,k)-designs are null (t',k)-designs."

3. Lower bound of the support size of null designs

Henceforth, for our main theorems, we assume that P is a finite ranked meet-semilattice (see [5]) with the condition (D_l) for some l.

Notation 3.1. Let $\omega = \sum_{x \in X_i^P} c_x x \in \mathbb{R}[X_i^P]$.

1. We define the *support* of ω by $Supp(\omega) \equiv \{x : c_x \neq 0\}$.
2. For $j \leq i$ and $y \in X_j^P$, $c_{\geq y} \equiv \sum_{y \leq x} c_x$.

Let μ_P be the Möbius function defined on P. The following is the key lemma for our main theorems, which can be found in [1] for the case of the Boolean algebras.

Lemma 3.2. *Let us assume that* $\omega = \sum_{x \in X_k^P} c_x x \in N_P(t,k)$. *Then for any fixed* $y \in X_{t+1}^P$, *and any* $z \leq y$, *the following holds*

$$\sum_{\substack{x \in X_k^P \\ x \wedge y = z}} c_x = c_{\geq y} \mu(z,y). \tag{3.1}$$

Proof. For $z \leq y$, let us consider the sum

$$S \equiv \sum_{z \leq z' \leq y} c_{\geq z'} \mu(z,z'). \tag{3.2}$$

By the definition of null t-designs and the condition (D_l), $c_{\geq z'} = 0$ for all $z' < y$. So,

$$S = c_{\geq y} \mu(z,y).$$

On the other hand,

$$\begin{aligned} S &= \sum_{z \leq z' \leq y} \left(\sum_{z' \leq x \in X_k^P} c_x \right) \mu(z,z') \\ &= \sum_{x \in X_k^P} \left(\sum_{z \leq z' \leq x \wedge y} \mu(z,z') \right) c_x = \sum_{\substack{x \in X_k^P \\ x \wedge y = z}} c_x \end{aligned}$$

since

$$\sum_{z \leq z' \leq x \wedge y} \mu(z,z') = \begin{cases} 0 & \text{if } z \neq x \wedge y \\ 1 & \text{if } z = x \wedge y \end{cases}$$

by definition of the Möbius function. This completes the proof. ∎

Theorem 3.3. *Let us assume that* $\omega = \sum_{x\in X_k^P} c_x x \in N_P(t,k)$ *but* $\omega \notin N_P(t+1,k)$. *Then there must be some* $y \in X_{t+1}^P$ *such that* $c_{\geq y} \neq 0$. *For any such fixed* y,

$$|Supp(\omega)| \geq |\{z \,:\, z \leq y,\ \mu(z,y) \neq 0\}|\,.$$

Proof. Since $c_{\geq y} \neq 0$, the right hand side of the equation (3.1) is nonzero and integral as long as $\mu(z,y) \neq 0$. Hence, for each $z \leq y$ such that $\mu(z,y) \neq 0$, there must be at least one $x \in Supp(\omega)$ such that $x \wedge y = z$. ∎

Let us think about the case $k = t+1$, i.e. an element $\omega \in N_P(t,t+1)$. Then y in Lemma 3.2 has rank $t+1 = k$, so $c_{\geq y} = c_y$ and (3.1) becomes

$$\sum_{\substack{x\in X_k^P \\ x\wedge y = z}} c_x = c_y \mu(z,y) \quad \text{for} \quad z \leq y. \tag{3.1$'$}$$

Theorem 3.4. *If* $\omega \in N_P(t,t+1)$, $\omega \neq 0$, *then*

$$|Supp(\omega)| \geq \min_{y\in X_{t+1}^P} \left(\sum_{z\leq y} |\mu(z,y)| \right).$$

Proof. Let $\omega = \sum_{x\in X_{t+1}^P} c_x x \in N_P(t,t+1)$, and let us assume that

$$\left|c_y\right| = \max\{|c_x| \,:\, x \in X_{t+1}^P\}.$$

Then we may assume that $c_y = 1$ and $|c_x| \leq 1$ for all $x \in X_{t+1}^P$, since we are only interested in the support size of ω. Hence (3.1$'$) is now

$$\sum_{\substack{x\in X_k^P \\ x\wedge y = z}} c_x = \mu(z,y) \quad \text{for} \quad z \leq y. \tag{3.1$''$}$$

Note that the values of the Möbius function are integers. Since $|c_x| \leq 1$ for all $x \in X_{t+1}^P$, for each $z \leq y$, there must be at least $|\mu(z,y)|$ many $x \in X_{t+1}^P$ such that $x \wedge y = z$ and $c_x \neq 0$. So the size of $Supp(\omega)$ is at least $\sum_{z\leq y} |\mu(z,y)|$. ∎

The following is immediate from the proofs of Lemma 3.2 and Theorem 3.4.

Corollary 3.5. *If the lower bound in Theorem 3.4 gives the tight bound for a finite ranked poset* P, *then the coefficients of minimal elements of* $N_P(t,t+1)$ *are* $\pm c$ *or* 0 *for some non-zero constant* c. *Moreover, for each* $y \in Supp(\omega)$ *and* $z \leq y$, *where* $\omega \in N_P(t,t+1)$ *is a minimal null design, there must be exactly* $|\mu(z,y)|$ *many* $x \in Supp(\omega)$ *such that* $x \wedge y = z$ *and* $c_x = sign(\mu_P(z,y)) c_y$. ∎

Remark 3.6.

1. The equation (3.1) is the key equation we used to prove Theorem 3.4, which simplifies when $k=t+1$. For general k, Theorem 3.3 is the best we can get from the equation (3.1), which does not seem to be as powerful as Theorem 3.4. We, however, will see some examples for which the lower bound given in Theorem 3.3 is tight.
2. Corollary 3.5 is a very powerful tool to understand the minimal null t-designs. For example, we were able to characterize the minimal null t-designs of some posets using Corollary 3.5.

Definition 3.7. If a finite ranked poset P satisfies the condition that for any $y, y' \in X_l^P$,

$$\sum_{z \le y} |\mu(z,y)| = \sum_{z' \le y'} |\mu(z',y')|,$$

then we say that P satisfies the condition (S_l).

Corollary 3.8. *If P satisfies the condition (S_{t+1}), then for $\omega \in N_P(t,t+1)$, $\omega \neq 0$,*

$$|Supp(\omega)| \ge \sum_{z \le y} |\mu(z,y)| \quad \text{for any} \quad y \in X_{t+1}^P. \qquad \blacksquare$$

We refer to Stanley [5, Chap. 3] for the definitions of the terms we use in the following corollary.

Corollary 3.9. *Let P be a finite ranked lower semimodular lattice which satisfies the condition (D_l) with its rank function ρ. Assume that the values of the Möbius function are non-zero and let n be the maximum of the values of ρ. If $n<k+t+1$, then $N_P(t,k)$ is the zero vector space.*

Proof. Suppose that there is a nonzero element $\omega = \sum_{y \in X_k} c_y y \in N_P(t,k)$. Then there must be $y \in X_s$, $s \ge t+1$ such that $c_{\ge y} \neq 0$ and for all $z \le y$, (3.1) holds. Hence, there must be $x \in Supp(\omega)$ such that $x \wedge y = \hat{0}$. This implies that $\rho(x \vee y) \ge \rho(x) + \rho(y) - \rho(x \wedge y) = s+k \ge k+t+1$ and $x \vee y$ is in P. This contradicts the fact $n<k+t+1$. $\blacksquare$

4. Examples

4.1. The subset lattice (Johnson scheme)

Let B_n be the Boolean algebra of $[n]=\{1,\ldots,n\}$, then B_n satisfies the condition (SD_l) for any $0 \le l \le n$. Note that $\sum_{z \le y} |\mu_{B_n}(z,y)| = \sum_{j=0}^{l} \binom{l}{j} = 2^l$, for a fixed

$y \in X_l^{B_n}$. Hence we have a lower bound 2^{t+1} for the support size of null t-designs. Moreover, $(x_1-x_2)(x_3-x_4)\cdots(x_{2t+1}-x_{2t+2})x_{2t+3}\cdots x_{k+t+1}$ is an element of $N_{B_n}(t,k)$ whose support size is 2^{t+1}, when we identify a subset $\{i_1,i_2,\ldots,i_l\}$ of $[n]$ with the product $x_{i_1}x_{i_2}\cdots x_{i_l}$. So, we have

Theorem . . *The minimum of the support size of nonzero null t-designs of B_n is 2^{t+1}.* ∎

4.2. The subspace lattice (q-Johnson scheme)

Let $L_q(n)$ be the lattice of subspaces of an n-dimensional vector space V over the Galois field $\mathbb{F}_q$, ordered by inclusion. Then $L_q(n)$ satisfies the conditions (SD_l) and (S_l) for any $0 \le l \le n$. Remember that the Möbius function on $L_q(n)$ is given by

$$\mu_{L_q(n)}(y,x) = (-1)^{i-j} q^{\binom{i-j}{2}} \quad \text{where} \quad x \in X_i^{L_q(n)},\ y \in X_j^{L_q(n)},$$

see [5, 3.10]. Therefore, by Theorem 3.4, the minimum of the support size of nonzero null t-designs in $N_{L_q(n)}(t,t+1)$ is at least

$$\sum_{j=0}^{t+1} \binom{t+1}{j}_q q^{\binom{t+1-j}{2}} = 2(1+q)(1+q^2)\cdots(1+q^t),$$

where $\binom{l}{m}_q$ is the q-binomial coefficient (see [5]).

Moreover, G. James explicitly gave a null $(t,t+1)$-design whose support size is $\prod_{i=0}^{t}(q^i+1)$. Note that $L_q(n)$ is a lower semimodular lattice (actually, it is modular), so we may assume that $k+t+1 \le n$ because of Corollary 3.9. Let W be a $2(t+1)$-dimensional subspace of V equipped with a bilinear form of the Gram matrix $\begin{pmatrix} \mathbf{0} & I_{t+1} \\ I_{t+1} & \mathbf{0} \end{pmatrix}$.

Proposition . . *$\sum_{W'}(sign\,W')W'$ is a null $(t,t+1)$-design whose support size is $\prod_{i=0}^{t}(q^i+1)$, where the sum is taken over all $(t+1)$-dimensional isotropic subspaces W' of W and the $sign\,W'$ is defined as $(-1)^{\dim(J\cap W')}$ for a fixed $(t+1)$-dimensional isotropic subspace J of W.* ∎

Hence, we have a proof of the conjecture of G. James on the support size of a certain representation of the general linear group over $\mathbb{F}_q$ [3, Chap. 18].

Theorem 4.3. *The minimum of the support size of non-zero elements of $N_{L_q(n)}(t,t+1)$ is $\prod_{i=0}^{t}(q^i+1)$.* ∎

5. Remarks

1. We applied the main theorems to many more ranked posets. For example, for the poset of dual polar spaces (see [6]), it turned out that the bound $\prod_{i=0}^{t}(q^i+1)$ in Theorem 3.4 is tight for types B_N, D_N, D_N^2, whereas the given bound is not tight for type C_N.
2. R. Liebler and K. Zimmermann [4] gave the characterization of the minimal null t-designs of the Boolean algebra when $k=t+1$. We characterized the minimal null t-designs of B_n when $n=k+t+1$. Moreover, we were able to give the characterization of the minimal null t-designs of $L_q(n)$, when $k=t+1$.
3. We believe that the minimal support size of non-zero null t-designs of $L_q(n)$, for any $t<k$, is also $\prod_{i=0}^{t}(q^i+1)$.

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